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Infinity subtraction in a quantum field theory of charges and monopoles†

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Abstract. Subtraction of ultraviolet infinities in Zwanziger's local quantum field theory of charges and monopoles is described. It involves an infinite number of graphs. The whole programme rests on the assumption that the infinite summations involved do not give rise to pathological situations and the Ward identities are satisfied even after the string cancellations. The resulting finite theory is Lorentz invariant.

1. Introduction

About ten years ago Zwanziger (1971) presented a local Lagrangian density for the quantum field theory of charges and monopoles. Unfortunately, as always happens with Dirac monopoles, this Lagrangian density depends on an arbitrary but fixed four-vector, say n_μ, the remnant of the old 'Dirac string' (Dirac 1948). Brandt *et al* (1979) gave a proof that full Green functions of gauge-invariant operators are n-independent, if Dirac's (1931) quantisation condition

$$e_i g_j/4\pi = \tfrac{1}{2}n_{ij} \qquad n_{ij} = 0, \pm 1, \pm 2, \ldots$$

between the electric and magnetic charges e_i, g_j is satisfied. A gauge-invariant regularisation of this quantum field theory also exists, in the presence of which the proof of n independence can be repeated (Panagiotakopoulos 1982).

As has been emphasised (Panagiotakopoulos 1982) the existence of these regulators guarantees the n independence of the finite Green functions that might result after any reasonable subtraction, provided that such a subtraction scheme can be found. By reasonable subtraction we mean any subtraction that does not introduce an n dependence in the finite subtracted Green functions. The purpose of the present work is to investigate the existence of such a subtraction.

The whole discussion is carried out in the context of the standard LSZ reduction formula which has been used as a guide to the correct renormalisation of the parameters of the theory. Also, the result of n independence of the renormalised S matrix is only valid if the standard reduction is used. However, it should be emphasised that the LSZ formulation does not take into account the long-range nature of the electromagnetic interaction. This long-range nature, if not taken care of properly, manifests itself in infrared divergences. The long-range monopole–electron interaction is also known (Zwanziger 1972) to give rise to an n-dependent phase factor which reflects

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the non-scalar nature of the S matrix in this theory. At this stage we can only hope that our treatment of the ultraviolet problem will not be invalidated by a complete treatment of the infrared one (Blagojevič and Senjanovič 1981, Blagojevič *et al* 1982). If a complete treatment of the long-range interaction does give rise to the n-dependent phase factor mentioned above this will not invalidate our claim of Lorentz invariance of the theory since this phase is not observable.

In the use of the LSZ formulation the assumption of the existence of free asymptotic states is inherent. This means that we assume the existence of an n-independent pole (or branch point) in the full fermion propagators. (Indeed if such a pole does not exist the theory would contain only confined states and so the study of scattering experiments would not make sense and our discussion would be futile.)

We will confine ourselves to the case of a spin-$\frac{1}{2}$ electrically charged particle with charge e and a spin-$\frac{1}{2}$ monopole with magnetic charge g. We will see that for the non-gauge-invariant fields it is the on-shell S matrix elements that are n-independent (Lorentz invariant). Then we will discuss the vacuum polarisation and charge, wavefunction and vertex renormalisation to all orders. Then a subtraction to all orders is described and the finiteness of the resulting theory is discussed. We also comment on the complications that arise when we renormalise the charges because of the quantisation condition. Later, a formal expansion in closed particle loops is described and its subtraction along the same lines is discussed. Finally we make some remarks stressing the assumptions made, the cases not discussed and possible work that can be done using a lattice version of Zwanziger's action that is proposed.

2. Definition of the theory

The manifestly local Zwanziger action with the inclusion of a gauge-fixing term and matter fields is

$$S = S_1 + S_2 \tag{2.1}$$

with

$$S_1 = -\tfrac{1}{2}\int d^4x\{[n\cdot(\partial\wedge A)]\cdot[n\cdot(\boldsymbol{\partial}\wedge B)^d]+[n\cdot(\partial\wedge A)]^2 - [\partial(n\cdot A)]^2+(A\rightarrow B, B\rightarrow -A)\} \tag{2.2}$$

and

$$S_2 = \sum_i \int d^4x\, \bar{\psi}_i(\mathrm{i}\not{\partial} - m_i - e_i\not{A} - g_i\not{B})\psi_i \qquad i=1,2 \tag{2.3}$$

where A_μ and B_μ are independent vector potentials, n_μ an arbitrary but fixed four-vector with $n^2=1$, and $\psi_1(\psi_2)$ is an electrically (magnetically) charged spin-$\frac{1}{2}$ matter field with electric (magnetic) charge $e_1=e$ ($g_2=g$), magnetic (electric) charge $g_1=0$ ($e_2=0$) and mass $m_1(m_2)$. The notation suppresses Lorentz indices, e.g., the scalar product of two four-vectors a^μ and b^μ is denoted $a\cdot b$, $a\wedge b$ means $a_\mu b_\nu - a_\nu b_\mu$ and for a second-rank tensor $F_{\mu\nu}$ the symbol $F\cdot F$ means $F_{\mu\nu}F^{\mu\nu}$. Also $F^d_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ is the dual of $F_{\mu\nu}$.

Variation of A and B gives the correct Maxwell equations and also the gauge-fixing equations

$$\partial^2 n\cdot A = \partial^2 n\cdot B = 0. \tag{2.4}$$

Variation of the charged fields gives the correct equations

$$(\mathrm{i}\not{\partial} - m_i - e_i \not{A} - g_i \not{B})\psi_i = 0. \tag{2.5}$$

With the notation $V^1_\mu = A_\mu$, $V^2_\mu = B_\mu$ and $\varepsilon^{12} = -\varepsilon^{21} = 1$, the free $(e_i = g_i = 0)$ propagators of the gauge fields can be written compactly in momentum space as (Zwanziger 1971)

$$D^{ab}_{\mu\nu}(k) \equiv \int \mathrm{d}^4x \, \exp(\mathrm{i}k \cdot x)\langle 0|TV^a_\mu(x)V^b_\nu(0)|0\rangle$$

$$= \left[\left(-g_{\mu\nu} + \frac{k_\mu n_\nu + n_\mu k_\nu}{n \cdot k}\right)\delta^{ab} - \frac{1}{n \cdot k}\varepsilon_{\mu\nu\rho\sigma} n^\rho k^\sigma \varepsilon^{ab}\right]\frac{1}{k^2 + \mathrm{i}\varepsilon}. \tag{2.6}$$

The non-covariant perturbation expansion that results from these propagators and the Feynman rules as read off directly from the Lagrangian can be shown to be unitary and consistent with the Faddeev–Popov formalism (Brandt and Neri 1978).

3. Proof of n independence of the S matrix

It is known that full Green functions of gauge-invariant operators, like the currents, are n-independent (Brandt *et al* 1979). The same is not true for full Green functions of gauge-non-invariant operators, for example, ψ_i and $\bar{\psi}_i$. What holds in this case is that the corresponding S matrix elements are n-independent. Details concerning the steps of the proof that are common to the one for Green functions of gauge-invariant operators can be found in the paper of Brandt *et al* (1979). Here we will only stress the differences.

We introduce into the generating functional of Green functions source terms for the spinor fields. It is essential for the proof that these source terms be gauge invariant. Such a gauge-invariant source is the path-dependent source (see, for example, 't Hooft 1976)

$$J^{\psi_i}(x)\left[\exp\left(\mathrm{i}a_i \int_{-\infty}^{x} \mathrm{d}x'_\mu \, V^i_\mu(x')\right)\right]\psi_i(x) \qquad (a_1 = e, a_2 = g) \tag{3.1}$$

where the integration is taken along a line from $-\infty$ to the point x. This source emits only single particles in lowest order and additional photons in higher order. These new couplings of the source contribute only to external line renormalisations and not to the S matrix. We integrate out the spinor fields in the generating functional $W_n(J^{\psi_i})$ and we express the result as a functional integral over classical particle paths. We make a change of variables from the Zwanziger classical particle action to the Schwinger–Yan non-local classical particle action. It is at this stage that the gauge invariance of the source is needed. Only if we integrate gauge-invariant quantities can the change of variables in the path integral be done without complications (Brandt and Neri 1978). After that we perform the change of integration variable $A \to A + \partial\lambda$, where $\partial\lambda$ is a singular function depending on n and such that the gauge transformation $A \to A + \partial\lambda$ in the Schwinger–Yan action is equivalent to the change $n \to n'$. Then we move the 'string' n to n'. Using the invariance of the $\mathrm{d}A$ measure under the change $A \to A + \partial\lambda$ we see that $W_{n'}(J^{\psi_i})$ differs from $W_n(J^{\psi_i})$ only in effective couplings of the source J^{ψ_i} with the gauge and spinor fields depending on n. When we cut the external legs, according to the standard reduction procedure, from the graphs that these new couplings give rise to, only those contributions survive which have a pole in the physical mass of the ψ_i. So the wavefunction renormalisation constant of ψ_i depends on n but not the S matrix. This in particular has as a consequence that when $\not{P} \to m_i$ (physical) the full ψ_i propagator has the form

$$\frac{Z_2(n)_i}{\not{P} - m_i} + \text{regular terms} \tag{3.2}$$

where the wavefunction renormalisation Z_{2i} depends on n. The situation is parallel to the gauge dependence of Z_2 in ordinary gauge theories.

The introduction of the regulators of Panagiotakopoulos (1982) does not change anything.

4. Regularisation

The easiest regulators (Panagiotakopoulos 1982) to work with give the (AA) and (BB) propagators multiplied by

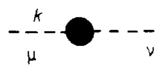
$$(1 + k^2/\Lambda^2 + k^4/\Lambda^4)^{-1}$$

where Λ is a cut-off parameter with the dimensions of mass and the mixed propagator, unregularised, in combination with Gupta regulators (auxiliary matter fields some of which obey the wrong statistics) for closed particle loops. They are sufficient. With these regulators everything we say can be repeated in their presence, since they do not introduce any complicated new n-dependence. The more complicated regulators that also regularise the mixed propagator are very difficult to work with; but since they render the theory finite and keep the proof of n independence their use presents no problem. With them of course the expressions that we will obtain will be much more complicated and extra spurious n dependence will appear. This extra n dependence, however, should not be regarded as invalidating the arguments based on n cancellation. These extra n-dependent terms disappear when the cut-off goes to infinity and obviously cannot cancel any n in the original theory (unless something pathological happens when we sum the infinite series of terms to all orders). So although we will present arguments using the unregularised form of the theory the presence of regulators will always be understood.

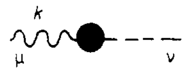
5. Vacuum polarisation

Let us first use the theorem of n independence of the full Green functions of the electric and magnetic current (J_μ and K_μ respectively) in order to find the form of the full gauge boson propagators. The full Green function of two electric currents $\langle T(J_\mu J_\nu)\rangle$ represented by

is of the form $i(g_{\mu\nu}k^2 - k_\mu k_\nu)C(k^2)$ using the gauge invariance and n independence to all orders. The full Green function of two magnetic currents $\langle T(K_\mu K_\nu)\rangle$ represented by

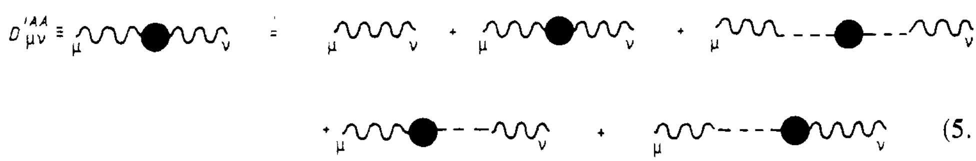


is of the form $i(g_{\mu\nu}k^2 - k_\mu k_\nu)D(k^2)$ for the same reasons. The same holds for the full Green function of one electric and one magnetic current $\langle T(J_\mu K_\nu)\rangle$ represented by



and having the form $i(g_{\mu\nu}k^2 - k_\mu k_\nu)F(k^2)$. The scalar functions C, D and F do not depend on n. The external legs are amputated everywhere.

The full (AA) propagator denoted $D'^{AA}_{\mu\nu}$ takes the contributions



(5.1)

where the external legs are not amputated. Because of current conservation the terms proportional to k_μ in the expressions of the blobs can be dropped. The last two terms cancel each other since the mixed propagator is antisymmetric in its free indices and even in k_μ (momentum), and $\langle T(J_\mu K_\nu)\rangle$ is symmetric in μ and ν and even in k_μ (momentum). The first three terms give

$$\begin{aligned} D'^{AA}_{\mu\nu} &= -\frac{g_{\mu\nu}}{k^2}(1+C(k^2)+D(k^2)) - \frac{n_\mu n_\nu - g_{\mu\nu}}{(n\cdot k)^2}D(k^2) \\ &= -\frac{g_{\mu\nu}}{k^2}(1+C(k^2)+D(k^2)) + \frac{T_{\mu\nu}}{(n\cdot k)^2}D(k^2) \end{aligned} \tag{5.2}$$

where $T_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$. Terms proportional to k_μ have been dropped. The same steps give

$$D'^{BB}_{\mu\nu} = -\frac{g_{\mu\nu}}{k^2}(1+C(k^2)+D(k^2)) + \frac{T_{\mu\nu}}{(n\cdot k)^2}C(k^2) \tag{5.3}$$

We see that the full (AA) and (BB) propagators are string dependent and not multiplicatively renormalisable. This makes obvious the fact that the theory cannot be perturbatively renormalisable in the strict sense, i.e. without introduction of counterterms of a different form from the terms originally present. The trouble in this case is that if we put even a finite number of counterterms of different form from the original terms the theory stops being Lorentz invariant to all orders. So unless the string-dependent terms effectively disappear the theory cannot be subtracted.

A problem related to the string dependence of the full (AA) and (BB) propagators is the definition of the screening of electric and magnetic charges due to vacuum polarisation. This screening is a physical process and should not be string dependent. The best candidate for the object corresponding to the usual Z_3 of QED is the $(1+C+D)$ multiplying the $g_{\mu\nu}$ term of both propagators at $k^2=0$. If we justify this choice properly we will have proved in a new, probably much easier, way the result obtained by Schwinger (1966) that the electric and magnetic charges acquire the same renormalisation due to vacuum polarisation. In fact Schwinger proved this result under a stronger quantisation condition for the charges (which would allow the change of the position of the string) and assuming that the string is not parallel to the line connecting them. Here, in contrast, we make use of the string independence of the theory.

Consider the sum of all graphs to all orders contributing to a physical process, e.g. e–e scattering, with all external legs cut according to the reduction formula, and on-shell. According to the theorem proved in § 3 this sum is n independent. Since we consider the sum of graphs to all orders we can replace every gauge boson propagator by a full propagator. Then each (AA) and (BB) propagator has the form given above. The string bit will cancel according to the theorem and the full propagator will be effectively proportional to $g_{\mu\nu}$. In fact this is correct, we believe, given that there seems to be no possibility of a remainder from the terms containing the combination $T_{\mu\nu}/(n\cdot k)^2$ surviving. This is so since there are no n_μ in the numerator anywhere in the expressions for the graphs except in terms containing a Levi–Civita tensor. But there, for each n in the numerator there is an n in the denominator and so no n excess. Two $\varepsilon_{\mu\nu\rho\sigma}n^\rho k^\sigma$ can be contracted, but in this case they give exactly the combination $T_{\mu\nu}/(n\cdot k)^2$. After all it is from this kind of contraction that this combination was initially created. One could argue that if we do not see $n_\mu n_\nu$ as being always associated with a $g_{\mu\nu}$, free n's exist in the numerator. Doing calculations of this kind we cannot reach any conclusion. We can only make the problem impossible. The correct way to see it is to recognise that $g_{\mu\nu}/(n\cdot k)^2$ cannot be cancelled leaving an n-independent remnant and, since $n_\mu n_\nu/(n\cdot k)^2$ always appears with $g_{\mu\nu}/(n\cdot k)^2$, it must be also cancelled completely. The only source of danger seems to be the summation of an infinite number of terms. We make what we believe to be a reasonable assumption that nothing pathological will happen when we sum all these terms. So the full (AA) and (BB) propagators are effectively the same which means that they are multiplicatively renormalisable.

We define $Z_3 = 1 + C(0) + D(0)$. We also define renormalised (AA) and (BB) propagators by

$$D'^{AA}_{R\,\mu\nu} = D'^{AA}_{\mu\nu}/Z_3 \qquad D'^{BB}_{R\,\mu\nu} = D'^{BB}_{\mu\nu}/Z_3 \tag{5.4}$$

and renormalised charges by

$$\sqrt{Z_3}e = e_R \qquad \sqrt{Z_3}g = g_R. \tag{5.5}$$

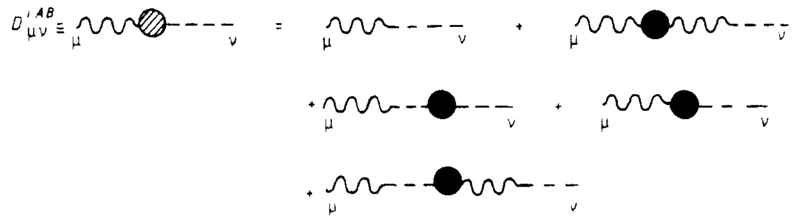
This amounts to defining a renormalised photon field by

$$\sqrt{Z_3}A^\mu_R = A^\mu \qquad \sqrt{Z_3}B^\mu_R = B^\mu. \tag{5.6}$$

We see that the renormalisation of these propagators can be absorbed into a renormalisation of the charges at the vertices connected through these propagators. Both charges are renormalised in the same way due to vacuum polarisation.

Once the contribution of vacuum polarisation to charge renormalisation has been defined in such a way as to absorb the photon-field renormalisation, the renormalisation of the mixed propagator is also bound to be the same as that of the (AA) and (BB) propagators if we do not want to introduce counterterms that would spoil the Lorentz invariance. This is so since the A and B fields are essentially one field, the photon field. We turn now to the computation of the full mixed propagator in order to see whether this renormalisation is sufficient to remove the infinity of the residue of the pole at $k^2=0$.

The full mixed propagator (AB) denoted $D'^{AB}_{\mu\nu}$ includes the contributions



(5.7)

The three first terms give a contribution proportional to the tree mixed propagator

$$-\frac{\varepsilon_{\mu\nu\rho\sigma}n^\rho k^\sigma}{k^2(n\cdot k)}(1+C(k^2)+D(k^2)). \tag{5.8}$$

The last two terms are identically zero. The reason for that is that all graphs contributing to $F(k^2)$ have an odd number of mixed propagators and so an odd number of $\varepsilon_{\mu\nu\rho\sigma}n^\rho k^\sigma$ terms because of Furry's (1937) theorem. An odd number of $\varepsilon_{\mu\nu\rho\sigma}n^\rho k^\sigma$ cannot be contracted completely. One of them at least should survive the contractions. However, by the n-independence theorem, ε cannot appear in the final answer for $\langle T(J_\mu K_\nu)\rangle$. So $F(k^2)$ is identically zero†. Since $F(k^2)$ is zero the mixed propagator is multiplicatively renormalisable and has the exact correction in order to have a finite residue at $k^2=0$ after renormalisation. We see that we could have defined the photon field and charge renormalisation starting from the mixed propagator. We also see that as happens with the (AA) and (BB) propagators the mixed propagator renormalisation can be absorbed into equal renormalisation of the electric and magnetic charges at the vertices connected by the mixed propagator. If $F(k^2)$ were not zero a new term, $-T_{\mu\nu}F(k^2)/(n\cdot k)^2$, would contribute to the full mixed propagator.

We can now check that this definition of Z_3 is the correct one and is consistent with the reduction formula for cutting external photon legs of Green functions in order to obtain S matrix elements. We consider a process in which, among other things happening, an external photon is attached to a charged-particle line. The reduction formula, as far as the photon leg is concerned, takes the form

$$Z_3^{-1/2}V^{a\mu}_{\text{ext}}D^{-1\,ab}_{\mu\nu}\langle TV^{b\nu}\psi\bar{\psi}\ldots\rangle \tag{5.9}$$

where $V^1_\mu=A_\mu$, $V^2_\mu=B_\mu$, $D^{-1\,ab}_{\mu\nu}$ is the inverse propagator as a 2×2 matrix and V^a_{ext} is a wavefunction for the V^a field satisfying the equation

$$D^{-1\,ab}_{\mu\nu}V^{b\nu}_{\text{ext}}=0. \tag{5.10}$$

The solution can be taken to be of the form $\exp(-\mathrm{i}k\cdot x)f^{a\nu}$ with $f^{a\nu}$ orthogonal to n^μ and k^μ and f^1 orthogonal to f^2. f^1 and f^2 are non-zero only if $k^2=0$ so that A and B really describe photons. Because the action of V^a_{ext} on the inverse propagator creates a zero at $k^2=0$ only corrections to (AA), (BB) and (AB) propagators having a pole at $k^2=0$ can contribute to Z_3. This guarantees that the string-dependent terms proportional to $T_{\mu\nu}/(n\cdot k)^2$ in the full (AA) and (BB) propagators do not contribute to vacuum polarisation, which is consistent with the definition we gave before.

† This argument is due to J Strathdee.

6. Proper vertices and fermion propagators

We consider first the sum of proper vertex parts with an A-field external leg and two electrically charged external matter-field legs (figure 1). We regard it as embedded in a sum of larger graphs with all kinds of corrections to all orders and with their external legs cut and on-shell (figure 2). According to the theorem of n independence all string dependence from the latter sum of graphs drops. Since the sum of proper vertex parts of figure 1 contains all graphs to all orders we can replace all gauge boson propagators with full ones. We know that all n dependence of this full vertex will disappear as a part of the extensive cancellations that take place between the graphs contributing to figure 2. For example some n dependence in the vertex under consideration might cancel some n dependence in another part of the larger graph. With this justification we drop the $T_{\mu\nu}/(n \cdot k)^2$ terms in the full (AA) and (BB) propagators and the $\varepsilon_{\mu\nu\rho\sigma}n^\rho k^\sigma$ after all possible contractions of ε have taken place, since contractions of this kind can leave n-independent pieces. Now, after neglecting terms that should cancel, the vertex reduces to a sum of n-independent QED-like graphs and so, just as in QED, when $P' \to P$ with $P^2 = m^2$ (m is the mass of the fermion), they will have the form $G^{-1}\gamma_\mu$, where G is a constant.

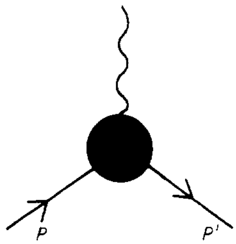


Figure 1.

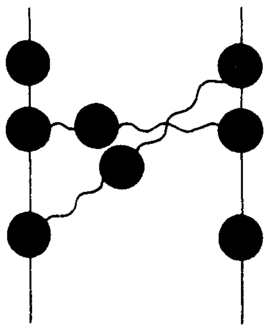


Figure 2.

Analogous things happen in connection with the full fermion propagator. In fact there is a correspondence between graphs contributing to the fermion proper self-energy and the proper vertex part before and after neglecting n dependence that will cancel. This correspondence is the usual Ward identity. When the fermion line is on-shell the full propagator behaves, neglecting n dependence that will cancel, like $H/(\not{P} - m)$. The Ward identity guarantees that $G = H$.

So in a sense the infinity of the vertex that survives the n cancellations will be cancelled by the infinities of the full nearby propagators that survive the n cancellations. In any on-shell sum of graphs to all orders and amputated according to the reduction formula the fermion propagators and the A-fermion–fermion vertices 'renormalise' each other and the infinities of the full gauge boson propagators are absorbed into charge renormalisation. This is exactly the situation in ordinary QED if we forget the lack of complications due to the string.

Of course we should remember that we have already defined the wavefunction renormalisation for the full fermion propagator. It is an n-dependent $Z_2(n)$ that we have used in order to cut the n-dependent external fermion legs according to the reduction formula. After these legs are cut the S matrix becomes n independent and the rest of the arguments are applicable. The two pictures that we have described, in external legs and inside the graph, approach each other if we adopt the following formal renormalisation definitions. The renormalised full ψ_1 propagator S'_{1R} is defined by $S'_{1R} = S'_1/Z_2(n)$ with S'_1 the unrenormalised full ψ_1 propagator and the renormalised full proper $A-\psi_1-\psi_1$ vertex $\Gamma^1_{\mu R}$ is related to the unrenormalised one Γ^1_μ by the relation $\Gamma^1_{\mu R} = Z_2\Gamma^1_\mu$. The charges do not get any new renormalisation beyond that coming from vacuum polarisation. In this way we cut the external legs correctly, we do not introduce any additional charge renormalisation as happened inside the graphs and we do not destroy the balance of infinities between the vertex and the nearby propagators.

These results could have been obtained formally by applying the Ward identity relating the proper vertex and the fermion proper self-energy while ignoring the n dependence. The result would be the old $Z_1 = Z_2$ relation of QED (Z_1 is the vertex renormalisation) which we saw to be effectively correct.

The $B-\psi_2-\psi_2$ vertex and ψ_2 propagator can be treated in exactly the same way.

Fermion mass renormalisation presents no problem. A renormalised mass can be defined by $m_R = m + \delta m$ and δm can be fixed such that the pole of the full fermion propagator be found in the right place.

A new complication that arises in this theory is the $A-\psi_2-\psi_2$ and $B-\psi_1-\psi_1$ vertices. These vertices do not exist in tree graphs but are created in higher orders. If these proper vertices are overall divergent and need an overall subtraction, the theory is in trouble since we have no counterterms for them. Fortunately, gauge and charge conjugation symmetries save it. Graphs contributing to the $B-\psi_1-\psi_1$ proper vertex parts are like those of figure 3.

The Ward identity relates the sum of these graphs when $P = P'$ and on-shell to the derivative of the graph of figure 4 which is zero by Furry's (1937) theorem. This means that we do not need an overall subtraction of this proper vertex at zero momentum transfer to the photon. These proper vertices will therefore be treated as skeletons.

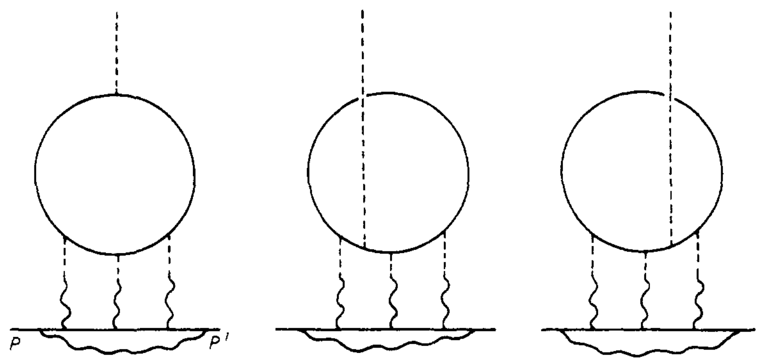


Figure 3.

7. Subtraction and finiteness

We consider the sum of all graphs to all orders contributing to a given physical process with the external legs cut and replaced with wavefunctions according to the reduction

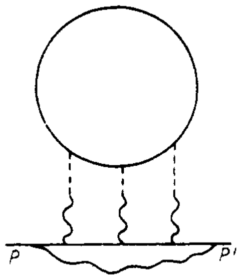


Figure 4.

formula and on-shell. We know that at this stage the n's cancel. We make the renormalisations described in the previous sections at the same time to all orders. The renormalisation constants are now multiple infinite power series in the coupling constants with coefficients depending on each other. In QED we get something like that to all orders but with the important difference that we can start from lower orders and construct the whole series in steps. Here we cannot say the same since at this stage we only know that the n cancellation occurs only when we sum an infinite series of graphs, and so we have to subtract the theory that results after this cancellation. Also in QED each graph either is primitively divergent and needs an overall subtraction, or is not and is finite given that in both cases all self-energy and proper vertex part insertions are made finite in a previous step. Here the difference is that there is no previous step. All steps have to be taken at the same time.

If we have a look at the Feynman rules of the Zwanziger theory we can see that they give rise to graphs with the same structure as in QED and with the same kind of infinities. In this theory we also have Ward identities of the same form as in QED. These two facts guarantee that one subtraction as described is enough to make the theory finite. The complication of this theory is due to the fact that we have, in finite orders, infinities that need counterterms with tensor structure different from the terms present in the Lagrangian, because of the n dependence, and not to the presence of worse (let us say quadratic) infinities. After the n cancellation counterterms of this kind are no longer needed.

8. Lorentz invariance of the subtracted theory

According to the renormalisation program discussed in the previous sections we renormalise each vector boson propagator by dividing with an n-independent constant Z_3, and both charges by multiplying the bare ones with $\sqrt{Z_3}$. Each fermion propagator is renormalised by dividing with an n-dependent constant $Z_2(n)$. Multiplication with the same constant renormalises the vertex of two fermion lines and one line of the vector field coupling to this fermion. There is also a mass counterterm.

Among all these renormalisation constants only the wavefunction renormalisation of the fermions is n-dependent. This should be so in order to remove the n-dependent external leg corrections and leave the S matrix n-independent. By renormalising the corresponding vertices, multiplying them with the inverse of this n-dependent constant, we make sure that the whole subtraction scheme does lead to a Lorentz-invariant subtracted theory.

9. Charge renormalisation and Dirac's quantisation condition

With the definitions $\sqrt{Z_3}e = e_R$ and $\sqrt{Z_3}g = g_R$ we keep e_R and g_R finite and fixed and the whole cut-off dependence goes to the bare charges e and g. But the bare charges satisfy Dirac's quantisation condition

$$eg = 2\pi\hat{n} \qquad \hat{n} = 0, \pm 1, \pm 2, \dots . \tag{9.1}$$

The proof of n cancellation of the S matrix uses this condition in an unavoidable way, and the whole subtraction is based on this cancellation. We also want to keep this condition for the renormalised charges if we want the renormalised theory to give reasonable predictions (Dirac 1931). Both these requirements can be satisfied if the bare charges go to infinity as the cut-off goes to infinity. We believe that this is not an absurd assumption given our experience with QED. In this case we require that the renormalised charges satisfy the condition and we also leave the bare ones to go to infinity in such a way that they also satisfy the condition. This can be done if we introduce a discrete cut-off which is allowed to take only those values for which $eg/2\pi$ is an integer. In a sense we renormalise the integer present in Dirac's condition and let the bare $\hat{n}$ go to infinity through integer values. If it should happen, for some reason that we cannot explain, that the infinities cancel and the bare charges remain finite the theory would still make sense if Z_3 is a ratio of appropriate integers.

10. The expansion in closed fermion loops

Up to this point we have been considering sums of graphs to all orders because our subtraction program was based on the n cancellation. It is true nevertheless that the n cancellation theorem asserts something even stronger, namely that this cancellation takes place even if we do not sum all the graphs contributing to a given process to all orders, but only all graphs with a given number of closed fermion loops and all radiative corrections (Brandt *et al* 1979). Everything we have said using the sum of graphs to all orders carries through if we consider the sum of graphs with a fixed number of closed fermion loops and all radiative corrections. The only reason we did not do it from the beginning is that, in our opinion, the argument to all orders can be presented in an easier way. Of course now the contributions to the renormalisation constants will be restricted.

It should be noted that the expansion in closed fermion loops is not a reasonable one since the number of closed fermion loops does not have much to do with the size of the contribution. However, it is technically satisfying that each sector of the theory which is Lorentz invariant is subtractable.

11. Remarks

(1) If in the quasi-static limit of the charge–monopole scattering a low-energy theorem is true saying that the whole amplitude reduces to the graph with one mixed propagator exchanged between the physical (renormalised) charges, then, since the answer should be n-independent (like the whole amplitude), it must be exactly zero given that $F = 0$. This would be a prediction of the theory in agreement with the classical result that a static charge does not feel the magnetic field.

(2) In order to avoid misunderstandings we would like to stress again the assumptions we have made which we believe are very reasonable. (a) The infinite summation and the removal of the regulators are interchangeable, (b) the string-dependent terms proportional to $T_{\mu\nu}/(n \cdot k)^2$ in the full gauge boson propagators will cancel completely in spite of the infinite sums of graphs involved in the cancellation procedure and (c) the n cancellation respects the Ward identities.

(3) We have covered only the case of spin-$\frac{1}{2}$ particles with only one type of charge. The only difference, as far as we can see, that might arise from the inclusion of dyons is the F term in the full mixed propagator. This F would still be zero if independent electric and magnetic charge conjugation is a symmetry. We have not dealt with the spin-0 matter-field case.

(4) Everything we have discussed in this paper involves sums of an infinite number of graphs and so new techniques have to develop in order to extract any information from this theory. All the complications seem to arise from the presence of the string that breaks the manifest Lorentz invariance of the Zwanziger Lagrangian. Over the last few years much attention has been paid to the lattice formulation of gauge theories (Wilson 1974) which provides a means of non-perturbative treatment. The lattice formulation sacrifices explicit Lorentz invariance but this does not seem to be so important in our theory since it does not have it anyway. That is why we believe it is not so crazy to write down a lattice version of Zwanziger's action that yields the correct continuum Zwanziger action in the 'naive' limit of zero lattice spacing. It is possible that if the theory is solved on the lattice and the correct quantum continuum limit is taken using renormalisation group techniques, the full quantum theory will be string independent.

In the lattice version the gauge fields live on the links of the lattice. $An\nu$ means that the object A lives on the link between the site n and the site $n+\hat{\nu}$ where $\hat{\nu}$ is the unit vector in the ν direction. In our notation α is the lattice spacing and we have chosen the string as a unit vector along the time axis. We will confine ourselves to one fermion with both types of charge. We define the following symbols.

$$F_{n\mu\nu} = (A_{n+\hat{\mu},\nu} - A_{n\nu} - A_{n+\hat{\nu},\mu} + A_{n\mu})/\alpha$$

$$H_{n\mu\nu} = (B_{n+\hat{\mu},\nu} - B_{n\nu} - B_{n+\hat{\nu},\mu} + B_{n\mu})/\alpha$$

$$f_{n\mu\nu} = \alpha^2 eF_{n\mu\nu} \qquad h_{n\mu\nu} = \alpha^2 gH_{n\mu\nu} \qquad \tilde{f}_{n\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}$$

$$\hat{k} = 1/(8+2m\alpha).$$

The lattice action that we propose is

$$\begin{aligned} I = & -\frac{\mathrm{i}}{8eg}\sum_{n\mu} \sin(2f_{n0\mu})\sin(2\tilde{h}_{n0\mu}) + \frac{\mathrm{i}}{8eg}\sum_{n\mu}\sin(2h_{n0\mu})\sin(2\tilde{f}_{n0\mu}) \\ & + \frac{1}{4e^2}\sum_{n\mu}(1-\cos(2f_{n0\mu})) + \frac{1}{4g^2}\sum_{n\mu}[1-\cos(2h_{n0\mu})] - \sum_n \bar{\psi}_n\psi_n \\ & + \tfrac{1}{2}\hat{k}\sum_n\sum_\mu \bar{\psi}_n(1-\gamma_\mu)\psi_{n+\hat{\mu}}\exp(2\mathrm{i}\alpha eA_{a\mu}) \\ & + \tfrac{1}{2}\hat{k}\sum_n\sum_\mu \bar{\psi}_{n+\hat{\mu}}(1+\gamma_\mu)\psi_n\exp(-2\mathrm{i}\alpha eA_{n\mu}) \end{aligned}$$

$$+\tfrac{1}{2}\hat{k}\sum_n\sum_\mu \bar{\psi}_n(1-\gamma_\mu)\psi_{n+\hat{\mu}}\exp(2\mathrm{i}\alpha g B_{n\mu})$$

$$+\tfrac{1}{2}\hat{k}\sum_n\sum_\mu \bar{\psi}_{n+\hat{\mu}}(1+\gamma_\mu)\psi_n\exp(-2\mathrm{i}\alpha g B_{n\mu}).$$

This lattice action has the symmetries

(i) $\psi_n \to \exp(2\mathrm{i}y_n e)\psi_n \qquad \bar{\psi}_n \to \exp(-2\mathrm{i}y_n e)\bar{\psi}_n \qquad A_{n\mu} \to A_{n\mu} - (y_{n+\hat{\mu}} - y_n)/\alpha$

(ii) $\psi_n \to \exp(2\mathrm{i}z_n g)\psi_n \qquad \bar{\psi}_n \to \exp(-2\mathrm{i}z_n g)\bar{\psi}_n \qquad B_{n\mu} \to B_{n\mu} - (z_{n+\hat{\mu}} - z_n)/\alpha$

(iii) $\alpha e A_{n\mu} \to \alpha e A_{n\mu} + \pi$

(iv) $\alpha g B_{n\mu} \to \alpha g B_{n\mu} + \pi$.

In order to get the naive continuum limit we assume that when $\alpha \to 0$, $\psi_n \simeq (\alpha^3/2\hat{k})^{1/2}\psi(n\alpha)$ and $A_{n\mu} \simeq A_\mu(n\alpha)$ with $n\alpha = x$. We expand in powers of α keeping only terms up to order α^4 (higher powers will not survive the $\alpha \to 0$ limit) and using $\int \mathrm{d}^4x = \alpha^4\Sigma_n$ we get Zwanziger's action in Euclidean space.

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